Computing Isogenies between Montgomery Curves Using the Action of $(0, 0)$

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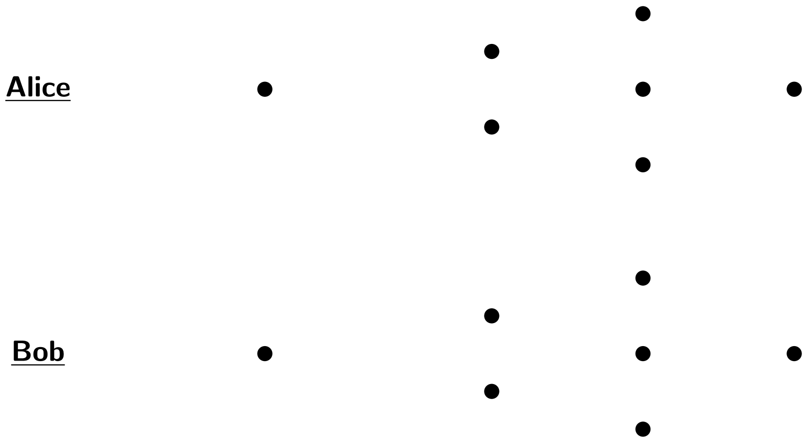
Supersingular isogeny-based cryptography

- Proposed by Jao & De Feo [JF11]
- Submitted to NIST competition [Aza+17] (*on Wednesday*)
  - SIDH (passive security)
  - SIKE (active security)
- This talk: *computing isogenies on curves with extra structure*

A graph-based protocol



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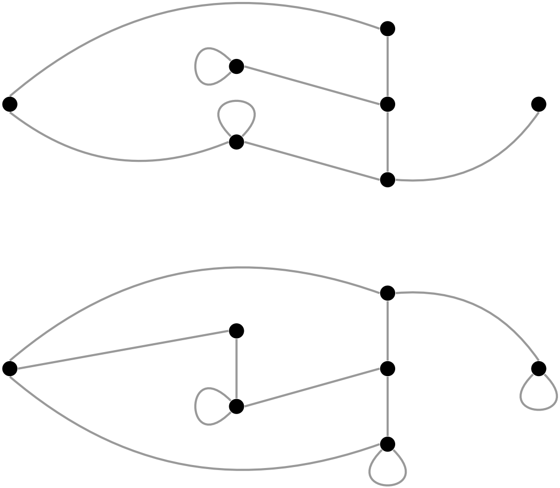

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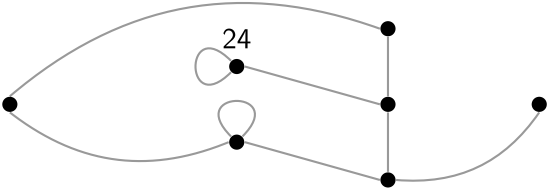

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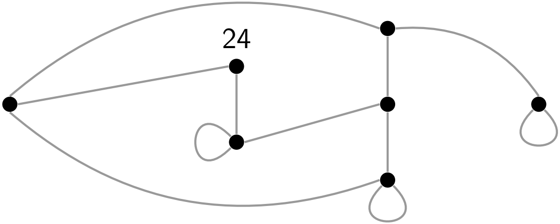

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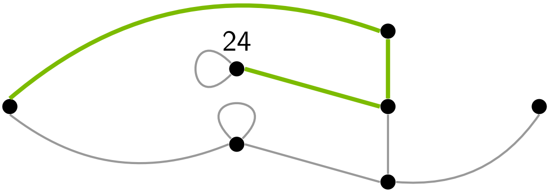

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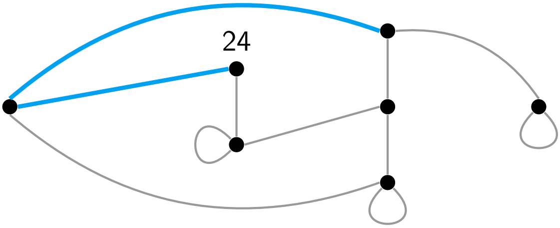

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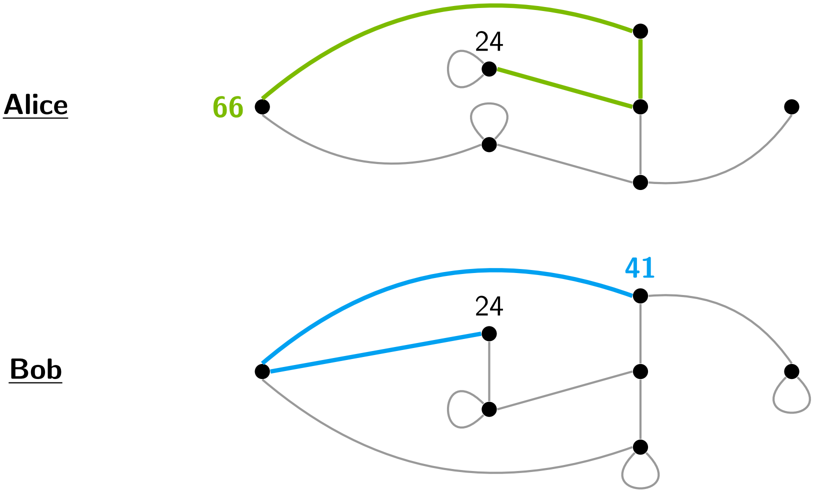

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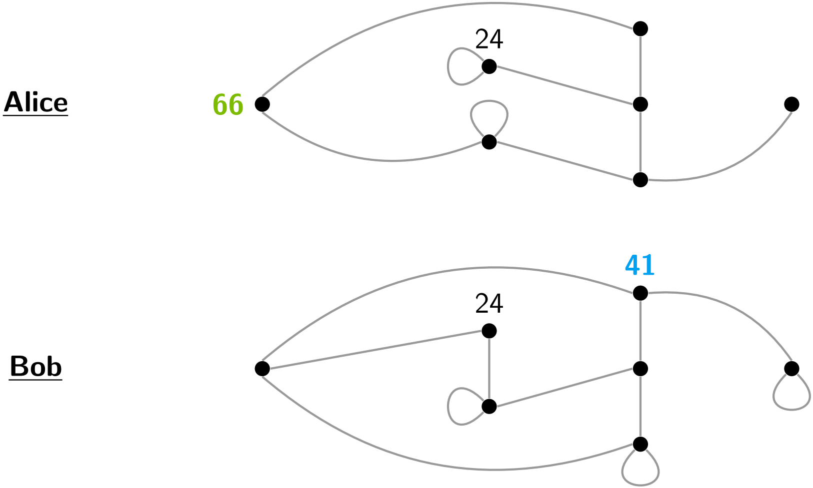

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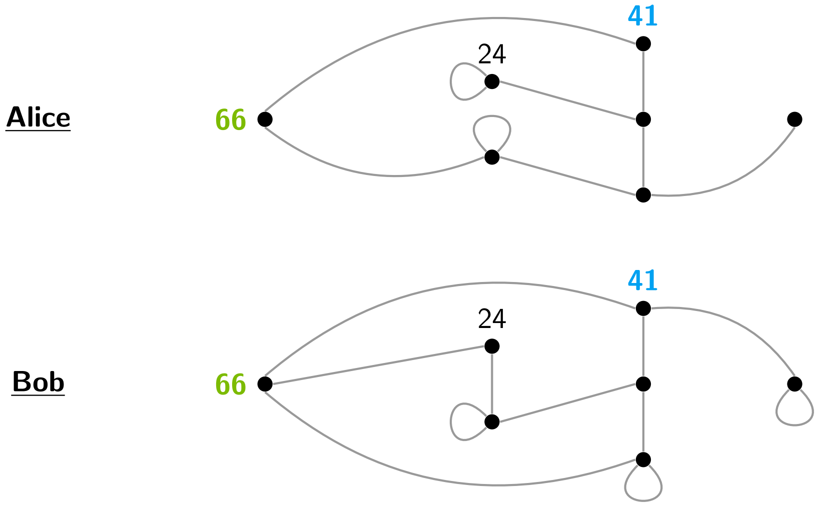

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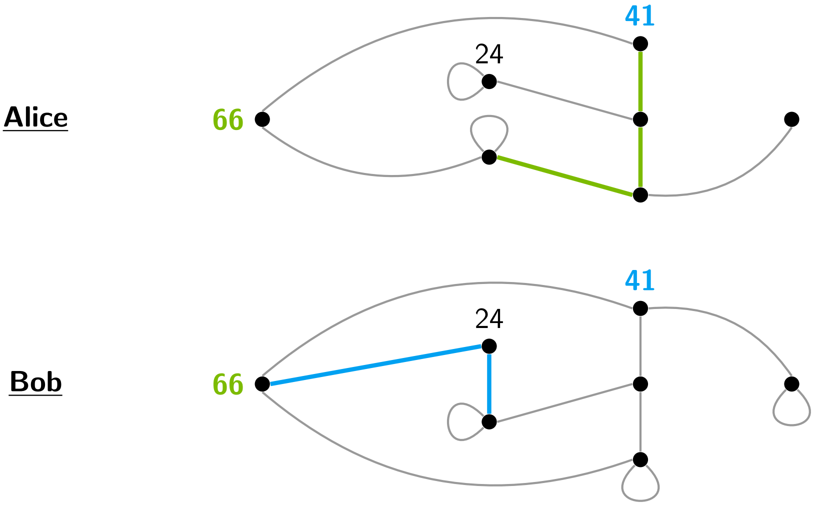

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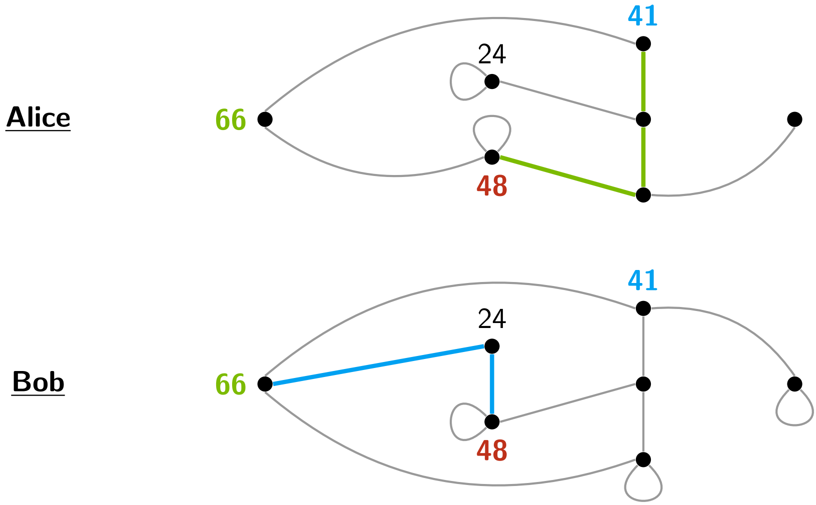
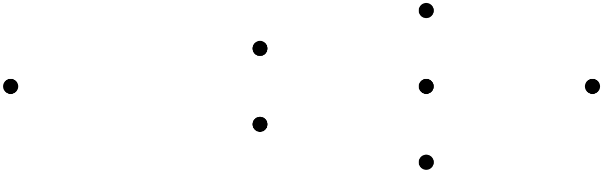


Constructing graphs and walks using isogenies



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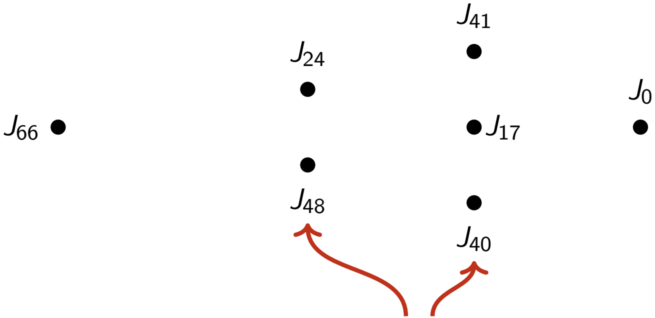
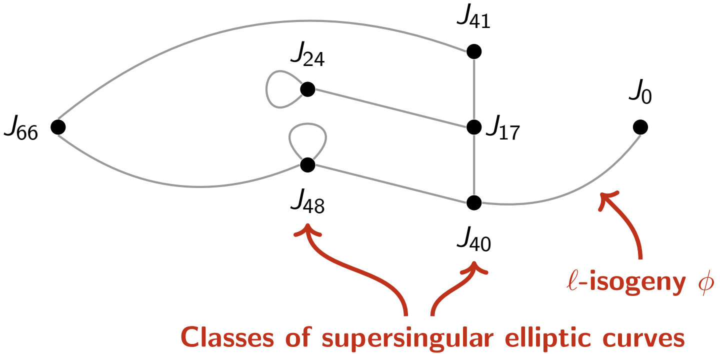


Classes of supersingular elliptic curves

Constructing graphs and walks using isogenies



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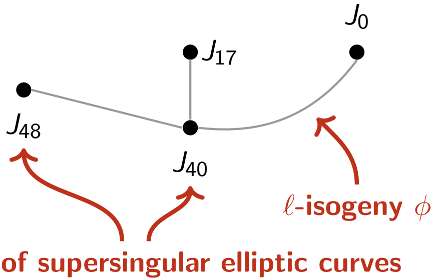
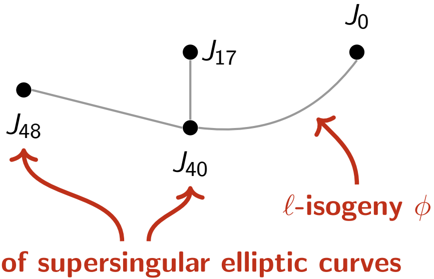


Constructing graphs and walks using isogenies

$$(1) \quad \Phi_\ell(X, Y) = X^{\ell+1} + Y^{\ell+1} + \cdots$$



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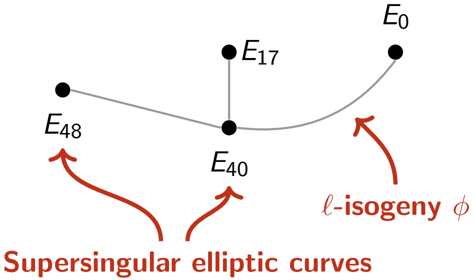
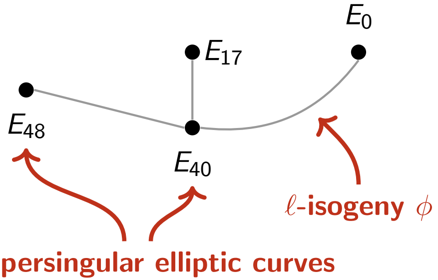


Constructing graphs and walks using isogenies

(1) $\cancel{\Phi_\ell(X,Y) = X^{\ell+1} + Y^{\ell+1} + \cdots}$

(2) $\ell + 1$ subgroups of order ℓ (Vélu's formulas)



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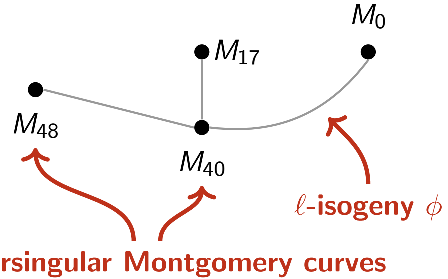

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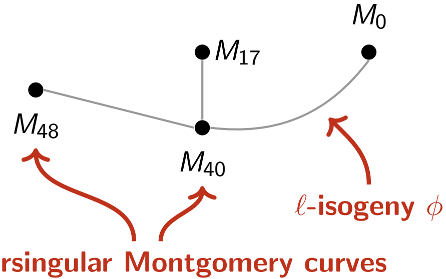

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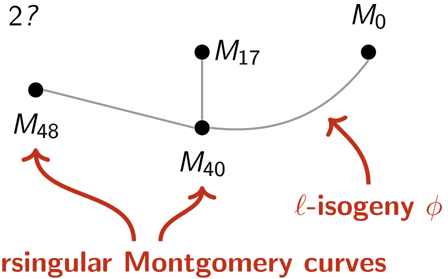
(Q1) *Where do these formulas come from?*

(Q2) *What about $\ell = 2$?*



What is an isogeny..

(1) A *morphism* of curves

$$M_A(x,y) = 0 \xrightarrow{\quad\quad\phi\quad\quad} M_{A'}(x,y) = 0$$

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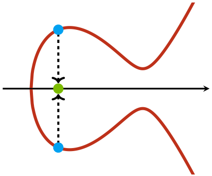

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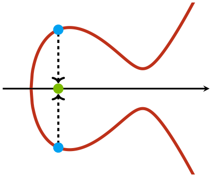

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(2) A *homomorphism* of groups

$$\begin{array}{ccccc} (x_0,—) & \oplus & (x_1,—) & = & (x_2,—) \\ \big\downarrow & & & & \\ \left(\frac{f(x_0)}{g(x_0)},—\right) & & & & \end{array}$$

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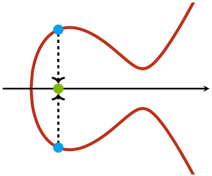

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Describing f and g

Given an isogeny $\phi(x) = \left(\frac{f(x)}{g(x)}, \text{---} \right)$

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Isogeny structure

Theorem (sketch)

Let $G \subset M(\bar{K})$ be a subgroup, $Q \notin G$ and

$$\phi = \left(\frac{f(x)}{g(x)}, - \right)$$

a separable isogeny such that $\ker \phi = G$ *and* $f(x_Q) = 0$. *Then*

$$f(x) = c_f \cdot \prod_{T \in G} (x - x_{T+Q}), \quad g(x) = \prod_{T \in G \setminus \infty} (x - x_T).$$

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- Generalizes when Q does not map to $(0, -)$
- Close connection between action of Q and isogeny!

Application to Montgomery curves

This works perfectly for Montgomery curves!

(1) A distinguished point $Q = (0, 0)$ of order two

(2) A very simple action $(x_T, —) + Q = \left(\frac{1}{x_T}, —\right)$

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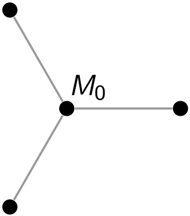
for *any subgroup* not containing $(0,0)$, generalizing [CH17]

Isogenies of degree two..

A curve has three points of order two, one of which is $(0, 0)$



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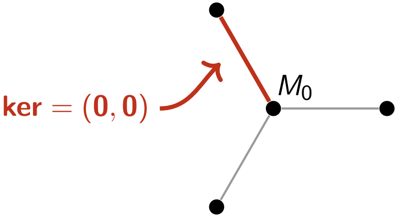

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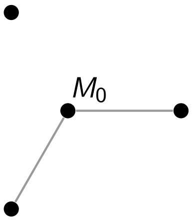

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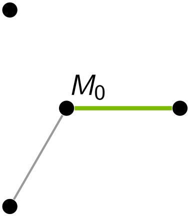

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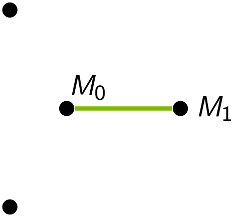

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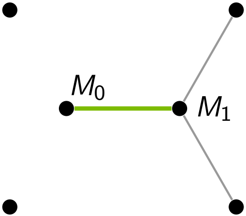

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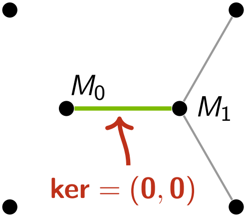

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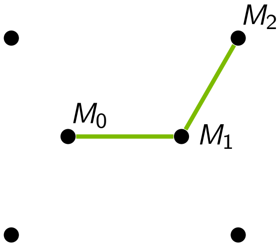

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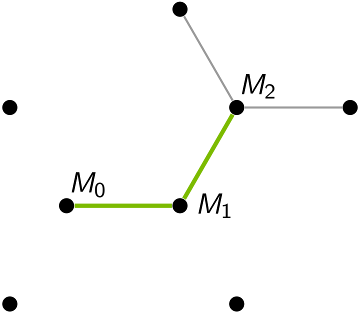

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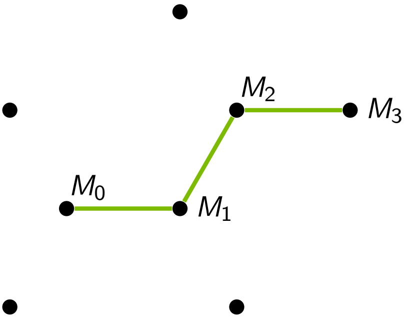

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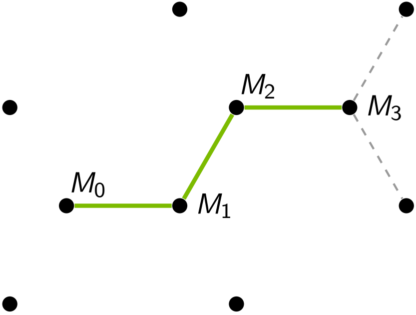

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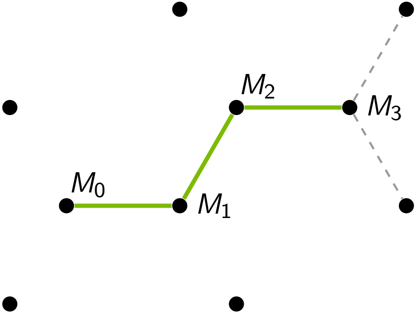

	2×2-isogeny	**4-isogeny**
Co-domain	$4\mathbf{S}$	$4\mathbf{S}+\mathbf{a}$

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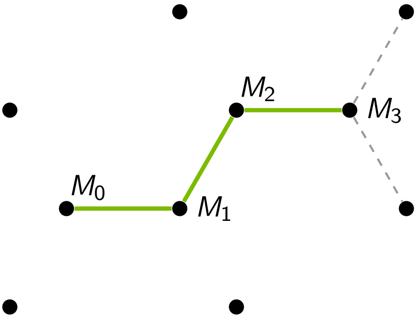

	2×2-isogeny	**4-isogeny**
Co-domain	$4\mathbf{S}$	$4\mathbf{S} + \mathbf{a}$
Evaluate	$8\mathbf{M} + 12\mathbf{a}$	$6\mathbf{M} + 2\mathbf{S} + 10\mathbf{a}$

Other curve models..

- Apply to *Tate Normal Form*
  - $y^2 + axy + by = x^3 + cx^2$
  - Point $Q = (0,0)$ of order ℓ
- For ℓ have $b = c = 0$ and

$$(x_T, y_T) + (0,0) = \left(\frac{-y_T}{x_T^2}, \frac{-y_T}{x_T^3} \right) .$$

  Results (currently) not better than Montgomery!
- Other models..?

Thanks for your attention!

`http://www.cs.ru.nl/~jrenes/`

References I

[Aza+17] Reza Azarderakhsh, Matthew Campagna, Craig Costello, Luca De Feo, Basil Hess, Amir Jalali, David Jao, Brian Koziel, Brian LaMacchia, Patrick Longa, Michael Naehrig, Joost Renes, Vladimir Soukharev and David Urbanik. *Supersingular Isogeny Key Encapsulation – Submission to the NIST's Post-Quantum Cryptography Standardization Process*. Available at `https://csrc.nist.gov/CSRC/media/Projects/Post-Quantum-Cryptography/documents/round-1/submissions/SIKE.zip`. 2017.

[CH17] Craig Costello and Hüseyin Hisil. "A Simple and Compact Algorithm for SIDH with Arbitrary Degree Isogenies". In: *Advances in Cryptology - ASIACRYPT 2017 - 23rd International Conference on the Theory and Applications of Cryptology and Information Security, Hong Kong, China, December 3-7, 2017, Proceedings, Part II*. 2017, pp. 303–329.

References II

[JF11] David Jao and Luca De Feo. "Towards Quantum-Resistant Cryptosystems from Supersingular Elliptic Curve Isogenies". In: *Post-Quantum Cryptography (PQCrypto 2011)*. Ed. by Bo-Yin Yang. Vol. 7071. Lecture Notes in Computer Science. Springer, 2011, pp. 19–34.