Improved Classical Cryptanalysis of SIKE in Practice

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Joost Renes Fernando Virdia

PKC 2020

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- Supersingular Isogeny Key Encapsulation (SIKE)
  $\Longrightarrow$ `https://sike.org/`
- Round-2 candidate in NIST standardization

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- **Today:** Further analysis of *classical* attacks on *SIKE*

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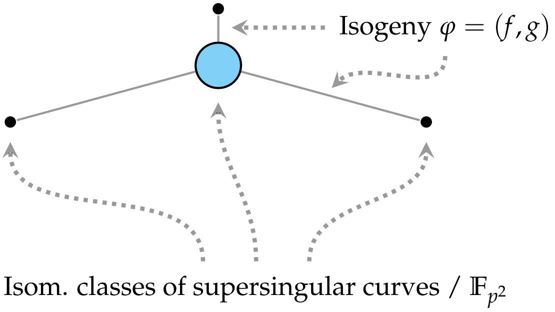
SIDH ($\ell = 2$)



(1) All classes of curves form a connected graph
(2) $\approx p/12$ nodes, each has $\ell + 1$ outgoing isogenies for prime ℓ

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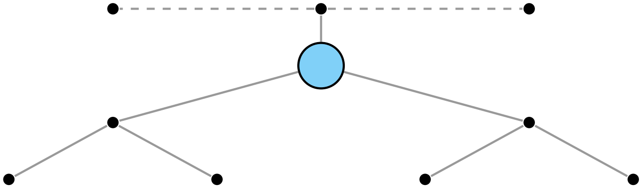

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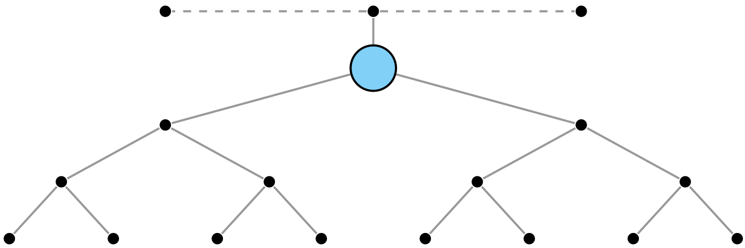

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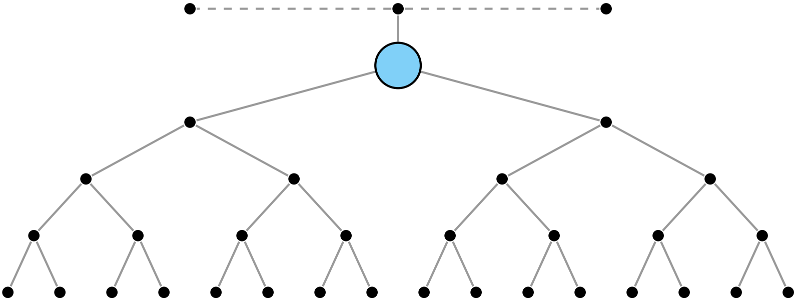

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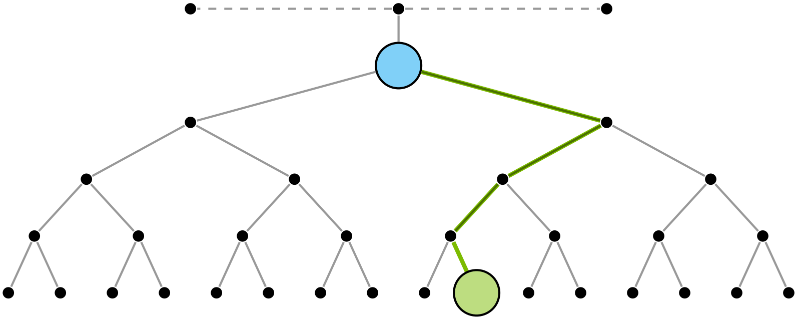

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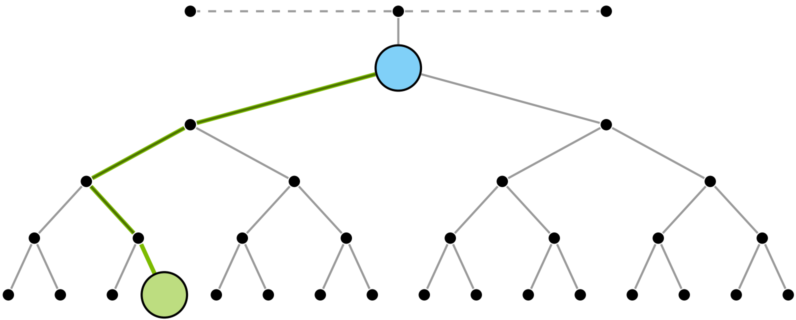

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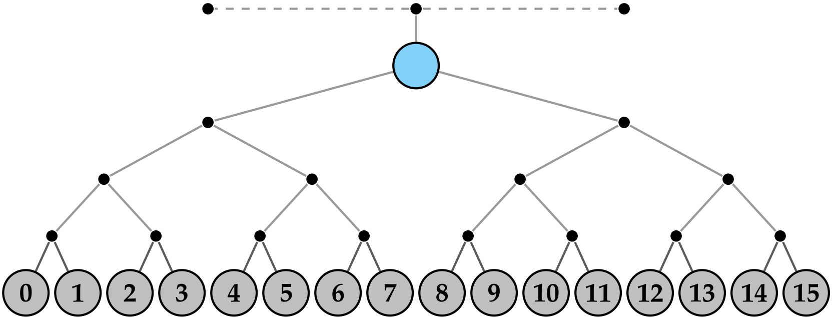

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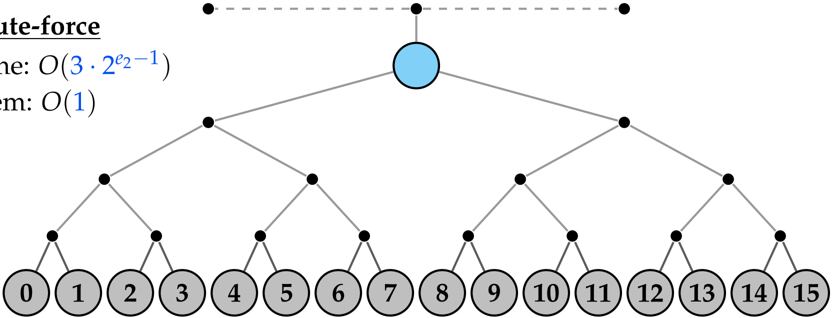
Brute-force

Time: $O(3 \cdot 2^{e_2 - 1})$

Mem: $O(1)$



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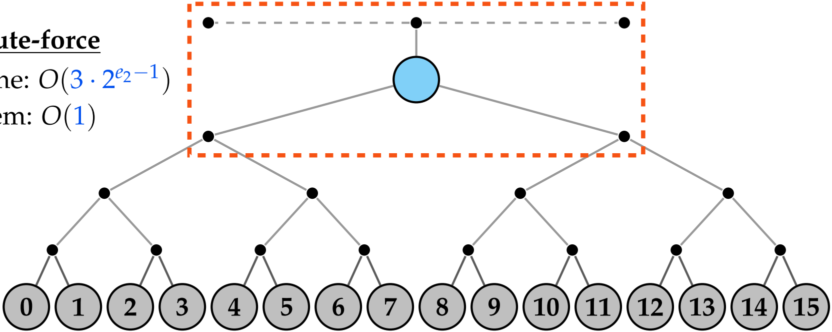

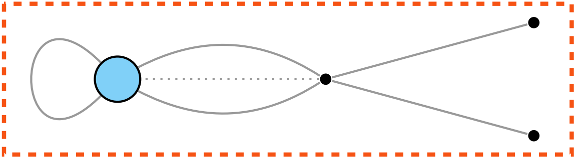

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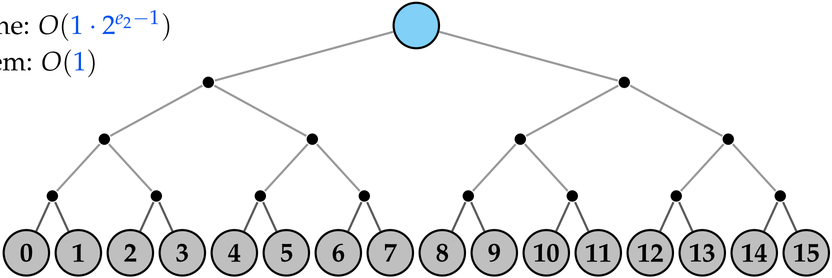

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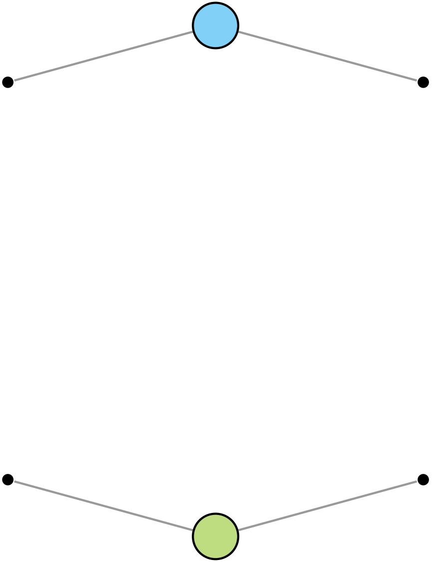

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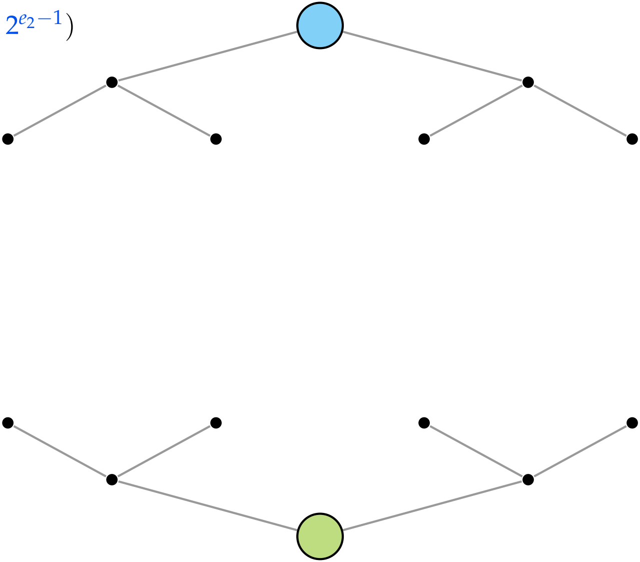

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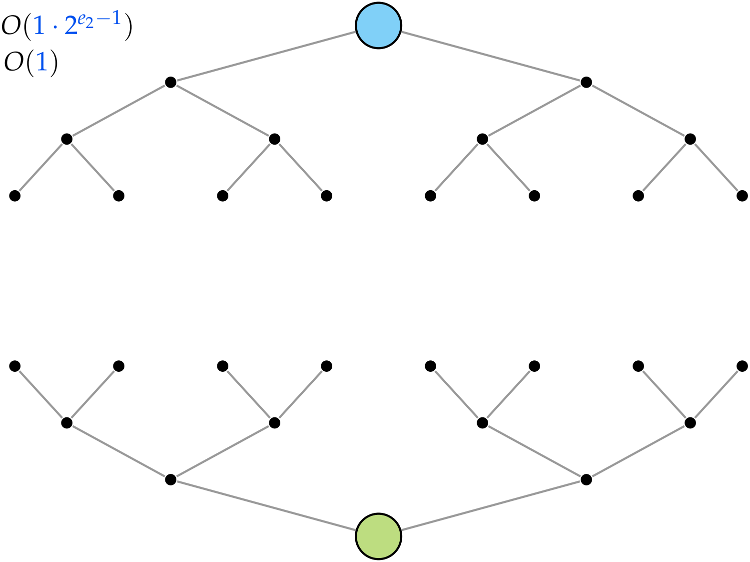

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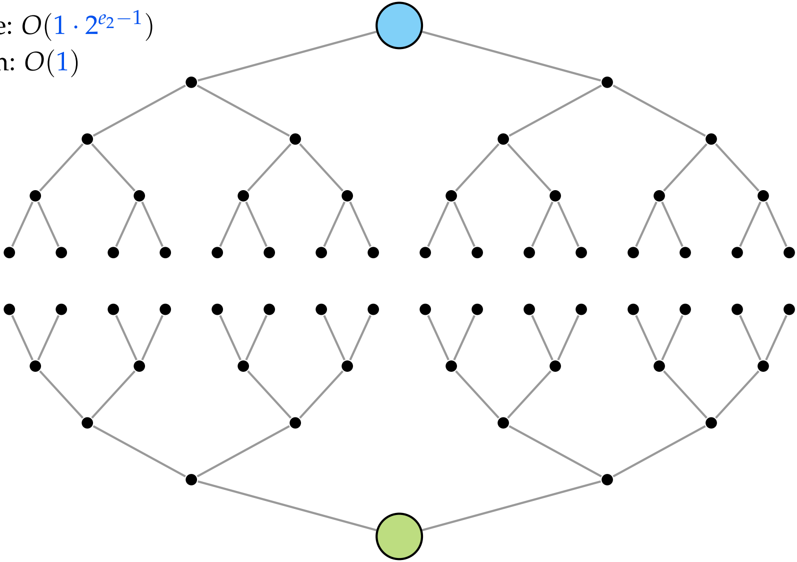

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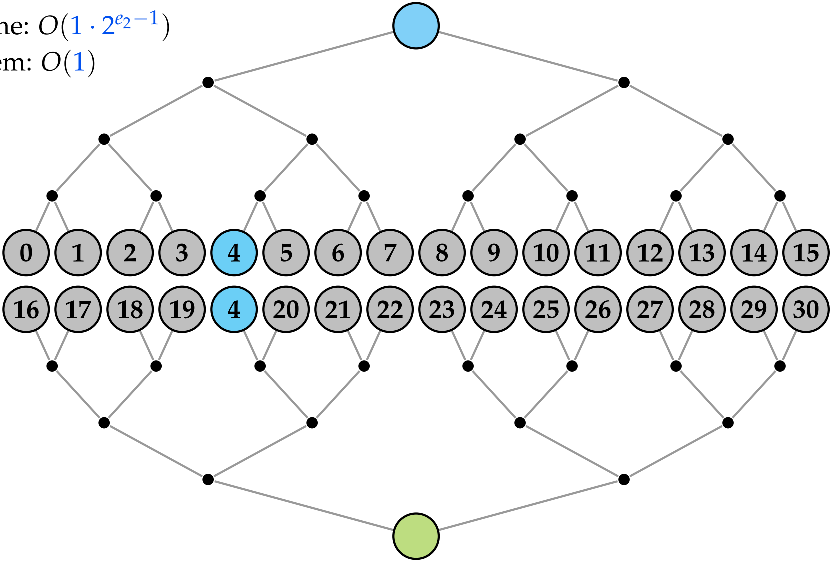

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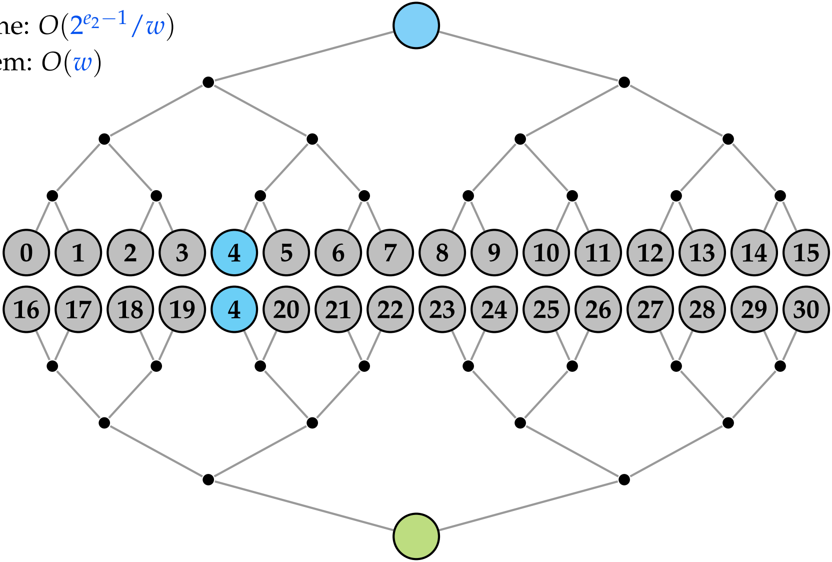
MitM

Time: $O(2^{e_2 - 1}/w)$

Mem: $O(w)$



SIKE ($\ell = 2$)

vOW

Time: $O(2^{3(e_2-1)/4}/\sqrt{w})$

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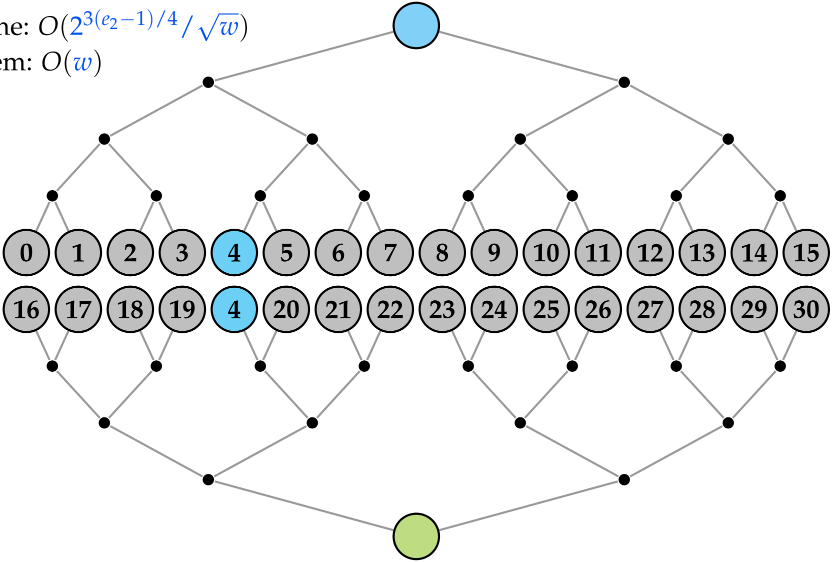

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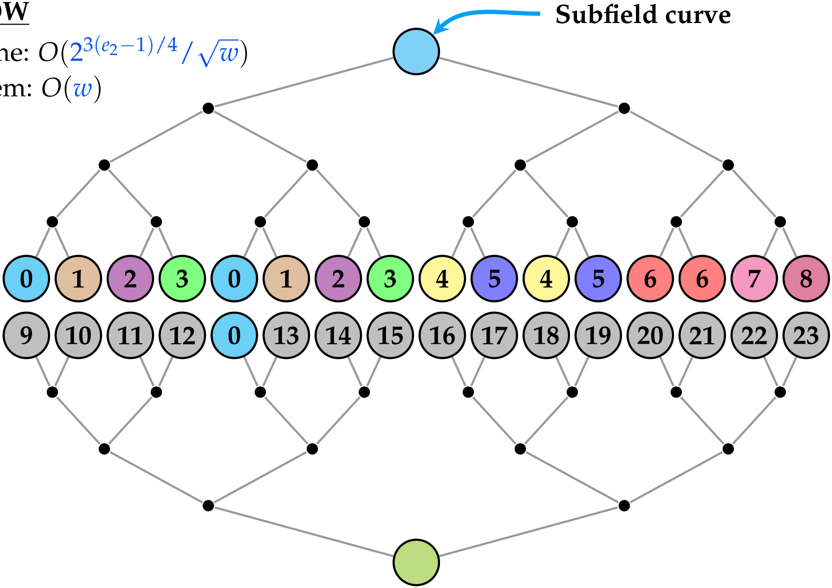

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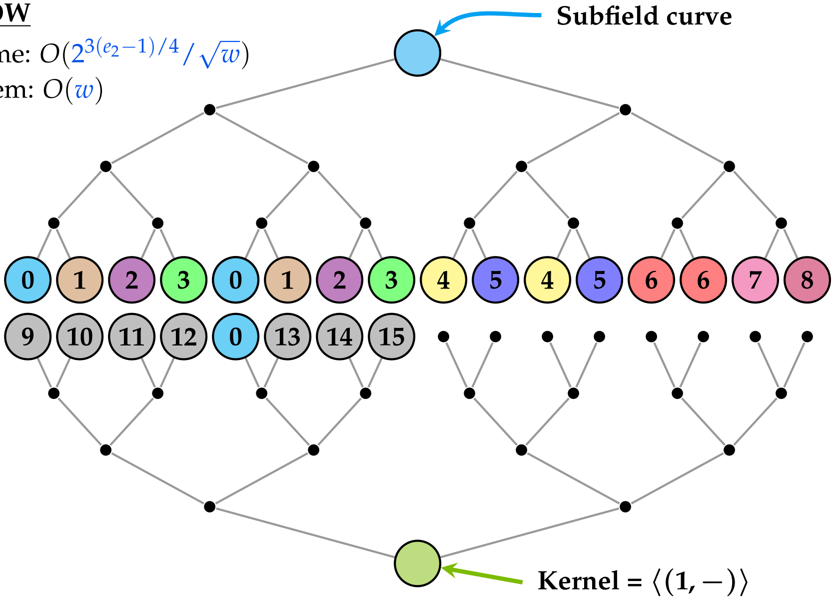

SIKE ($\ell = 2$)

vOW on SIKE

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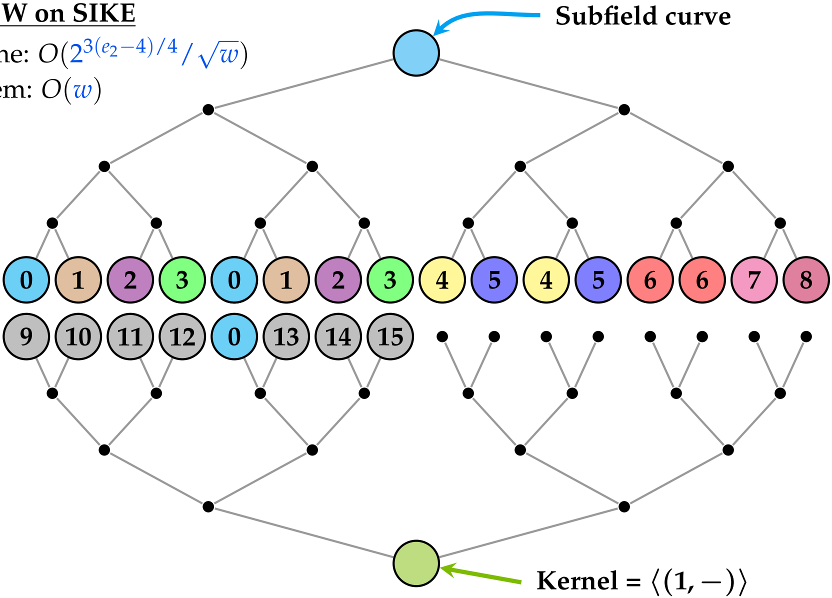

van Oorschot-Wiener

A set $S = \{0, \ldots, N-1\}$ and functions $h_0, h_1 : S \to T$

MitM problem

Find $x, y \in S$ such that $h_0(x) = h_1(y)$

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- Define a family $\{f_n\}_{n \in \mathbb{N}}$ of functions

$$f_n : S^* \to S^*, \quad S^* = S \times \{0,1\}$$
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- The $\{g_n\}_{n \in \mathbb{N}}$ are "random" (e. g. SHA-3 domain sep. n)

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$\implies$ MitM reduces to golden collision search in any of the f_n

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Given f_n (for fixed n) find the golden collision

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4. If new, store in memory and goto 2
5. If not new, check for golden collision. If not, store and goto 2
6. If found $10w$ distinguished points, try next n

van Oorschot-Wiener applied to SIDH/SIKE

The set $S = \{0, \ldots, \sqrt{2^{e_2}} - 1\}$ and (family of) random functions

$$f_n : z \in S^* \mapsto \text{AES-CBC}_n(j(E_i / \langle P_i + [z]Q_i \rangle))$$

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(Here E_0 is the *starting curve* and E_1 the *public key*.)

Application to SIDH / SIKE

e_2	$\log w$	$\log$ #**Queries** to f_n
		Exp. SIDH
32	9	23.20
36	10	25.70
40	11	28.20
44	13	30.20

[5] **canadians**.

Application to SIDH / SIKE

		log #**Queries** to f_n	
e_2	$\log w$	Exp.	**[canadians]**
		SIDH	SIDH
32	9	23.20	24.38
36	10	25.70	27.25
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SIKE parameter choices + Equivalence classes

$\Longrightarrow$ N decreases by factor 6

$\Longrightarrow$ $N^{3/2}/w$ decreases by factor $\approx$ 15

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$\implies N$ decreases by factor 6

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44	13	30.20	26.32	30.91	30.37	26.42
56	17	37.20	33.32	–	–	33.38

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From theory to practice

Appearing soon: `https://github.com/microsoft/vOW4SIKE`

- C library building on SIKE submission

$\Longrightarrow$ Fast arithmetic

$\Longrightarrow$ Speed-up oracle queries f_n

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  - $\Longrightarrow$ Complexity $O(2^{3(e_2-4)/4}/(m\sqrt{w}))$

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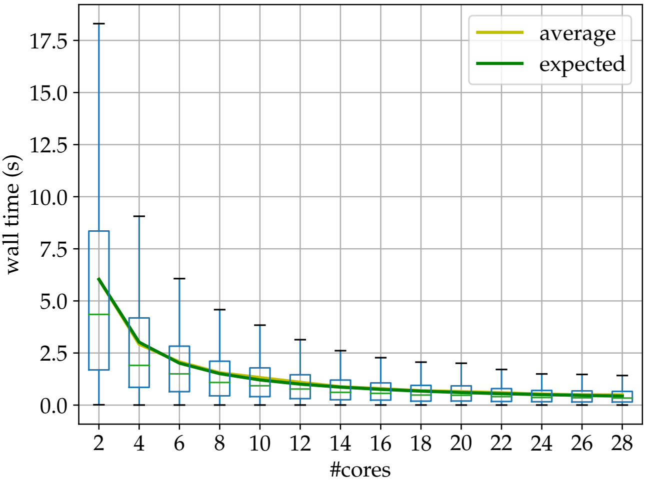
  - $\Longrightarrow$ Number of memory accesses becomes bottleneck
  - $\Longrightarrow$ Have to synchronize n across instances
  - $\Longrightarrow$ We built only *multi-core*, steps towards true *distributed*
  - $\Longrightarrow$ Instances have (small) *local memory*, use this

Parallelized van Oorschot-Wiener



[6]Experiments run on 2x Intel(R) Xeon(R) E5-2690 v4 at 2.60 GHz with 14 cores each

Local memory for checking collisions

Collision checking

$(z, \bar{z}, d)$ z ● ● $\bar{z}$

$(y, \bar{z}, e)$ y ● ● $\bar{z}$

Local memory for checking collisions

Collision checking

$(z, \bar{z}, d)$

$(y, \bar{z}, e)$

Local memory for checking collisions

Collision checking

$(z, \bar{z}, d)$

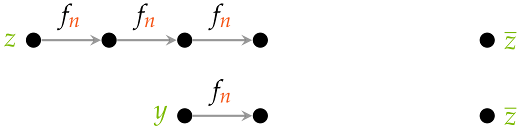
$(y, \bar{z}, e)$

Local memory for checking collisions

Collision checking

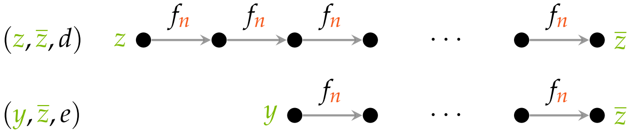
$(z, \bar{z}, d)$

$(y, \bar{z}, e)$



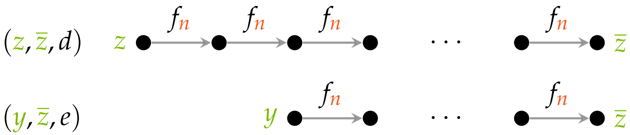
Local memory for checking collisions

Collision checking

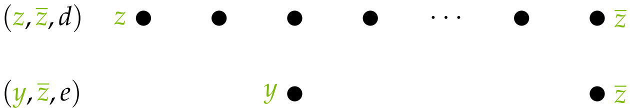


Local memory for checking collisions

Collision checking

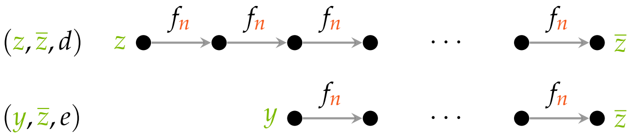


Additional assumption: can store *all* intermediate points locally

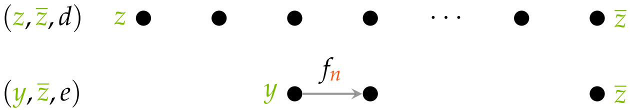


Local memory for checking collisions

Collision checking

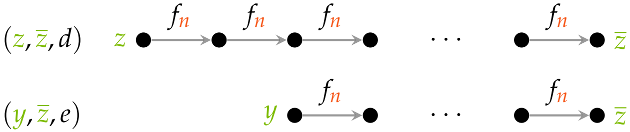


Additional assumption: can store *all* intermediate points locally

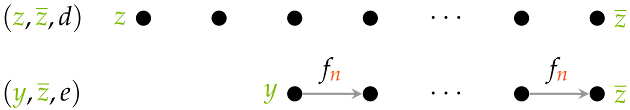


Local memory for checking collisions

Collision checking

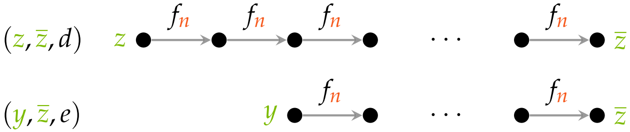


Additional assumption: can store *all* intermediate points locally

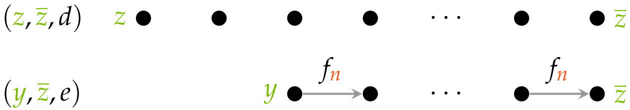


Local memory for checking collisions

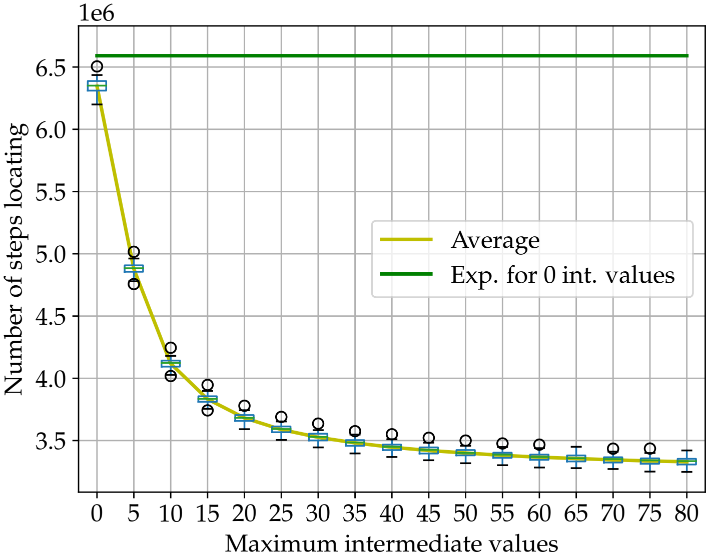
Collision checking



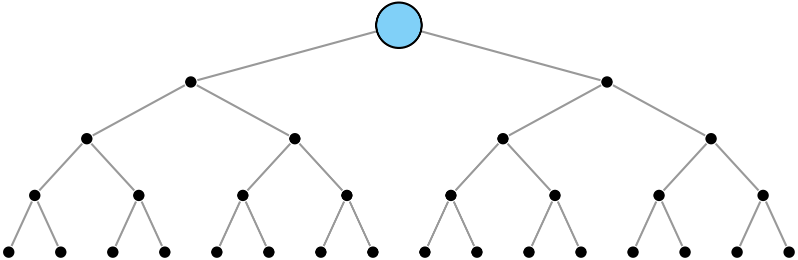
Additional assumption: can store t intermediate points locally



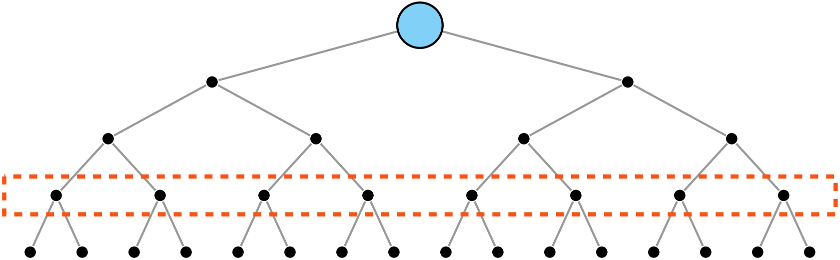
Local memory for checking collisions



Local memory for precomputation

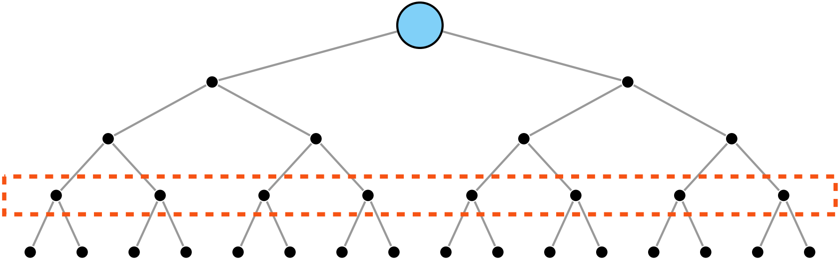


Local memory for precomputation



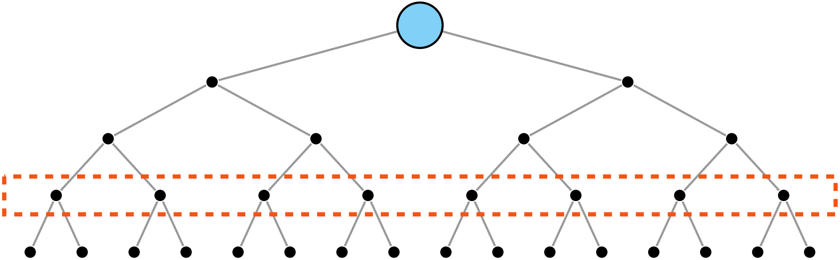
- Fix precomputation depth Δ (could vary per device)

Local memory for precomputation



- Fix precomputation depth Δ (could vary per device)
- Isogenies of degree $2^{2(e_2-1)-\Delta}$
- Table size of $2 \cdot 3 \cdot 2^{\Delta}$ elements in $\mathbb{F}_{p^2}$

Local memory for precomputation



- Fix precomputation depth Δ (could vary per device)
- Isogenies of degree $2^{2(e_2-1)-\Delta}$
- Table size of $2 \cdot 3 \cdot 2^{\Delta}$ elements in $\mathbb{F}_{p^2}$
- Make sure to store the right basis of $2^{e_2-\Delta}$-torsion..

Other optimizations

- Multi-target attacks
- Compressing distinguished points — “leading bits are zero”

Thanks!

$$\begin{array}{ccc} E_0 & \dashrightarrow & E_A \\ \downarrow & & \downarrow \\ E_B & \dashrightarrow & E_{AB} \end{array}$$